On the Least Prime Number in a Beatty Sequence

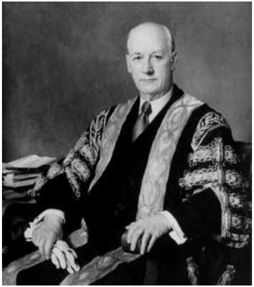
Jörn Steuding (Würzburg)
-- joint work with Marc Technau --
Salamanca, 31 July 2015

i. Beatty Sequences

ii. Prime Numbers in a Beatty Sequence

iii. The Least Prime in a Beatty Sequence

i. Beatty Sequences

Beatty sequences were first introduced by the physicist John William Strutt (Lord Rayleigh, on the left) in 1894. The name, however, is with respect to Samuel Beatty who popularized the topic by a problem he posed in 1926 in the *American Mathematical Monthly*.

Beatty Sequences

Denote by $\lfloor x \rfloor$ the largest integer $\leq x$.

Given a positive real number α, the set

$$\mathcal{B}(\alpha) = \{\lfloor n\alpha \rfloor \ : \ n \in \mathbb{N}\}$$

is called the associated Beatty sequence. For example,

$$\begin{aligned} \mathcal{B}(1) &= \{1, 2, \ldots\} = \mathbb{N}, \\ \mathcal{B}(\sqrt{2}) &= \{\lfloor 1.41\ldots \rfloor = 1, \lfloor 2 \cdot 1.41\ldots \rfloor = 2, \lfloor 3 \cdot 1.41\ldots \rfloor = 4, \ldots\} \end{aligned}$$

If α is rational, then $\mathcal{B}(\alpha)$ is a union of arithmetic progressions.

Surprisingly, *if α is irrational and β is defined by $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\mathcal{B}(\alpha) \cup \mathcal{B}(\beta) = \mathbb{N}$ is a disjoint union.*

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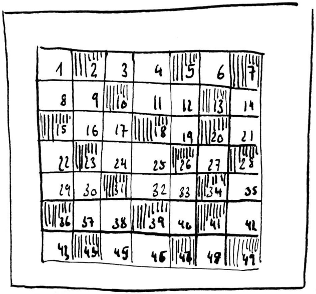
Examples

The partition of $\mathbb{N}$ with

$$\alpha = \frac{1}{2}(\sqrt{5}+1) \quad :$$



$$\begin{aligned}
B(\tfrac{22}{7}) &= \{3,6,9,12,15,18,\mathbf{22},25,28,31,34,37,40,\mathbf{44},47,50,\ldots\},\\
B(\tfrac{355}{113}) &= \{3,6,9,12,15,18,\mathbf{21},25,28,31,34,37,40,\mathbf{43},47,50,\ldots\},\\
B(\pi) &= \{3,6,9,12,15,18,\mathbf{21},25,28,31,34,37,40,\mathbf{43},47,50,\ldots\}.
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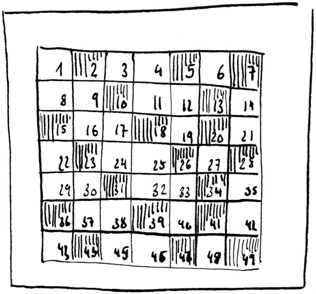

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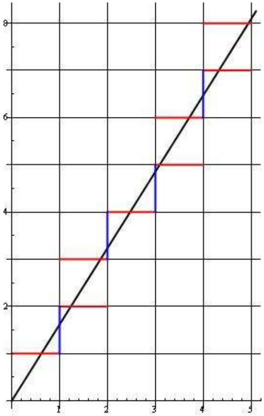
Geometry



The straight line

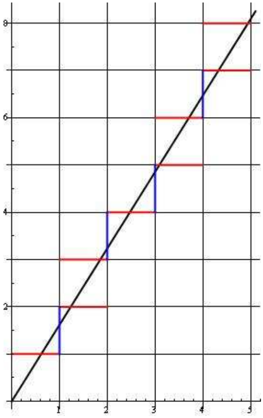
$$Y = \frac{1}{2}(\sqrt{5}+1)X$$

does not contain any rational points; its intersections with vertical lines are

$$1, 3, 4, 6, 8, \ldots,$$

that are the elements of the Beatty sequence $\mathcal{B}(\frac{1}{2}(\sqrt{5}+1))$.

The Sturmian word associated with $\frac{1}{2}(\sqrt{5}+1)$ is the infinite aperiodic sequence 010010100100 . . .



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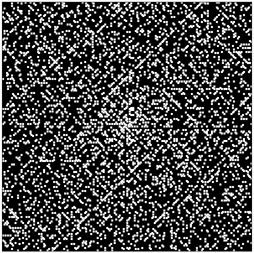
ii. Prime Numbers in a Beatty Sequence



The Ulam spiral shows the sequence of positive integers in a spiral, the primes colored in white the others in black. The picture reminds us of a galaxy in outerspace, indicating the random appearance of primes. What can be said about the intersection of the set of prime numbers and a Beatty sequence?

Primes in Arithmetic Progression

In 1837-39 Peter Gustav Lejeune Dirichlet proved that *every prime residue class* $a \bmod q$ *contains infinitely many primes.*

This may be used to find primes in a Beatty sequence $\mathcal{B}(\alpha)$ whenever α is rational, e.g.

$$\mathrm{B}(\tfrac{22}{7}) = \{3, 6, 9, 12, 15, 18, 22\} + 22\mathbb{N}_0.$$

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Prime Number Theorems

Dirichlet proved for the number $\pi(N; a \bmod q)$ of primes $p \leq x$ in the arithmetic progression $a + q\mathbb{Z}$ satisfying $\gcd(a, q) = 1$ that

$$\pi(N; a \bmod q) = \frac{1}{\varphi(q)}\pi(N) + \text{error term},$$

where $\varphi(q)$ counts the number of $a \leq q$ being coprime with q. This shows that the primes are equidistributed in the prime residue classes!

The celebrated prime number theorem from 1896 (proved by Jacques Hadamard and Charles de la Vallée Poussin) states

$$\pi(N) = \frac{N}{\log N} + \text{error term}$$

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Exponential Sums in Primes

In the process of proving the ternary Goldbach conjecture for sufficiently large odd integers Ivan Matveevich Vinogradov obtained in 1937/38, *for* α, a *and* q *satisfying*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2},$$

the estimate

$$\sum_{p \le N} \exp(2\pi i m \alpha p) \ll N^{1+\epsilon} \left(\frac{1}{N^{1/2}} + \frac{q}{N} + \frac{m}{q} + \frac{m^4}{q^2} \right)^{1/2} = o(\pi(N)),$$

where the summation is over all primes $p \le N$.

The Primes are Uniformly Distributed Modulo One

Vinogradov's estimate implies that the sequence of numbers αp with fixed irrational α is uniformly distributed modulo one, which means that in every interval $[a, b) \subset [0, 1)$ the proportion of fractional parts $\{\alpha p\}$ is equal to $b - a$ (that is the length of the interval).

This follows from a classical criterion of Hermann Weyl from 1913/14: *A sequence of real numbers x_n is uniformly distributed modulo one if and only if for every integer $m \neq 0$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \exp(2\pi i m x_n) = 0.$$

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Another Easier Example

The sequence of numbers αn with $n = 1, 2, \ldots$ is uniformly distributed modulo one if and only if α is irrational; this was first proved by Piers Bohl in 1909 (improving an old result of Leopold Kronecker).

The idea behind Weyl's criterion: the map

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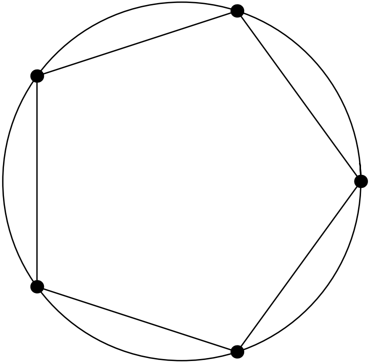
transforms the unit interval to the unit circle; the values of a uniformly distributed sequence add up to something small for all integers $m \neq 0$.



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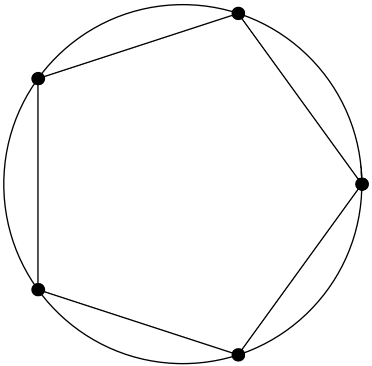

The Prime Number Theorem for Beatty Sequences

It follows from uniform distribution modulo one that the number $\pi_{\mathcal{B}(\alpha)}(x)$ of prime numbers $p \le x$ in a Beatty sequence $\mathcal{B}(\alpha)$ satisfies

$$\lim_{x\to\infty} \pi_{\mathcal{B}(\alpha)}(x) \cdot \frac{\alpha \log x}{x} = 1.$$

In particular, there are infinitely many primes in a Beatty sequence; more precisely:

$$\pi_{\mathcal{B}(\alpha)}(x) = \frac{1}{\alpha}\pi(x) + \text{error term};$$

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iii. The Least Prime in a Beatty Sequence

$$4,\ 21,\ 38,\ 55,\ 72,\ 89,\ 106,\ 123,\ 140,\ \ldots\ \in 4 + 17\mathbb{N}_0$$

The least prime in the arithmetic progression $1 + 5227\mathbb{N}_0$ is $397\,253$ and appears as $1 + 76 \cdot 5227$ (as computed by Will Jagy in MathOverflow, Nov 14, 2011).

The least prime number in the Beatty sequence $\mathcal{B}(\exp(\pi\sqrt{163}))$ is $3\,938\,061\,189\,611\,531\,159$ which appears with $n = 15$.

The Least Prime in an Arithmetic Progression

In 1944 Yuri Linnik showed that *for every sufficiently large q and coprime a, there exists a constant* expo *such that the least prime p in the residue class $a \bmod q$ satisfies*

$$p \leq \mathrm{const} \cdot q^{\mathrm{expo}}$$

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The question dates back to Savardaman Chowla (on the left) who conjectured in 1934 that one may even take $\mathrm{expo} = 1 + \epsilon$.

In 1937 Pál Turán proved that this is true i) for almost all q and, ii) in general under the assumption of the generalized Riemann hypothesis.

What to expect for the least prime in a Beatty sequence?

For $2 \leq m \in \mathbb{N}$ let

$$\alpha_m = 4 + \frac{\sqrt{2}}{m}.$$

Then

$$\lfloor n\alpha_m \rfloor = 4n \qquad \text{for} \quad n = 1, 2, \ldots, M := \left\lfloor \frac{m}{\sqrt{2}} \right\rfloor.$$

Hence, these numbers are divisble by 4 so the least prime p in $\mathcal{B}(\alpha_m)$ does not appear among the first M elements.

There cannot be any bound similar to the one in the rational case!

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A Theorem of Robert C. Vaughan (1977)

Let $\alpha, \beta \in \mathbb{R}$ and suppose that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$$

for coprime integers a and q. Define

$$\chi(\theta) = \begin{cases} 1 & \text{if } -\delta \le \{\theta\} := \theta - \lfloor\theta\rfloor < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any fixed $\delta \in (0, \frac{1}{2})$,

$$\begin{aligned} \sum_{n \le N} \Lambda(n)\chi(n\alpha - \beta) \;&=\; 2\delta \sum_{n \le N} \Lambda(n) + O\left(\delta^{\frac{2}{5}} N^{\frac{4}{5}} \left(\frac{Nq}{\delta}\right)^{\epsilon} + \right. \\ &\left. + \left(\log \frac{Nq}{\delta}\right)^{8} \left(\frac{N}{q^{\frac{1}{2}}} + N^{\frac{3}{4}} + (\delta N q)^{\frac{1}{2}}\right)\right). \end{aligned}$$

Our Application

Here $\Lambda(n)$ counts prime powers $n = p^k$ with weight $\log p$; hence the prime number theorem gives

$$\sum_{n \leq N} \Lambda(n) \sim \sum_{p \leq N} 1 \cdot \log N = \pi(N) \cdot \log N \sim N.$$

Thus, we get a similar formula for primes only!

Combining this and Vaughan's formula with $\delta = \alpha^{-1}$ we aim at finding

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Continued Fractions

Choosing $N \approx q^{1+\epsilon}$ finally leads to the equivalent inequality

$$1 > \mathrm{constant} \cdot \alpha^{\frac{1}{2}+\epsilon} q^{-\epsilon},$$

which is fulfilled for sufficiently large q. If α is irrational, there exist indeed infinitely many reduced fractions $\frac{a}{q}$ with

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}.$$

For example, one may take the convergents $\frac{p_n}{q_n}$ to the continued fraction expansion of α which are bounded below by the Fibonacci numbers:

$$q_n \geq F_{n+1} \approx \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}+1}{2}\right)^{n+1}.$$

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This implies that

there exists a positive integer n such that for every irrational $\alpha > 1$ the least prime $p \in \mathcal{B}(\alpha)$ satisfies

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Further Questions for Future Research

- Try to improve the estimate $p \leq q_n^{1+\epsilon}$. What can be said under assumption of the generalized Riemann hypothesis?

- What about the error term in the asymptotic formula for the number of primes in a Beatty sequence:

$$\left|\pi_{\mathcal{B}(\alpha)}(N) - \tfrac{1}{\alpha}\pi(N)\right| < ???$$

Can you improve on Vinogradov's bounds?

- Is the (ternary) Goldbach conjecture true for primes from a Beatty sequence?

Formulate your own statement about primes in a Beatty sequence that your neighbour cannot answer! And prove it!!

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¡ muchas gracias !